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CYCLOTOMIC UNITS IN $\mathbb{Z}_p$-EXTENSIONS

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ABSTRACT

Let K_0 be the maximal real subfield of the field generated by the p-th root of 1 over $\mathbb{Q}$, and K_∞ be the basic $\mathbf{Z}_p$-extension of K_0 for a fixed odd prime p. Let K_n be its n-th layer of this tower. For each n, we denote the Sylow p-subgroup of the ideal class group of K_n by A_n, and that of E_n/C_n by B_n, where E_n (resp. C_n) is the group of units (resp. cyclotomic units of K_n. In section 2 of this paper, we describe structures of the direct and inverse limits of B_n. The direct limit, in particular, is shown to be a direct sum of λ copies of p-divisible groups and a finite group M, where λ is the Iwasawa λ-invariant for K_∞ over K_0. In section 3, we prove that the capitulation of A_n in A_m is isomorphic to M for $m \gg n \gg 0$ by using cohomological arguments. Hence if we assume Greenberg's conjecture ($\lambda = 0$), then A_n is isomorphic to B_n for $n \gg 0$.

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1. Introduction

Fix an odd prime p. For each $n \geq 1$, let ζ_{p^n} be a primitive p^n-th root of 1 such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. Let $K_n = \mathbf{Q}(\zeta_{p^{n+1}} + \zeta_{p^{n+1}}^{-1})$ so that $K_\infty = \cup_{n\geq 0} K_n$ is the basic $\mathbf{Z}_p$-extension of K_0. Let E_n (resp. C_n) be the group of units (resp. cyclotomic units) of K_n. We denote the p-part of the ideal class group of K_n by A_n and that of E_n/C_n by B_n. There seem to be more conjectures than facts about the behaviour of A_n. For instance, Vandiver's conjecture says that $A_0 = 0$ and Greenberg's conjecture says that A_n is bounded as $n \to \infty$, or equivalently, the Iwasawa λ-invariant of K_∞/k_0 equals zero. It is even unknown whether or not the natural map $j_{n.m} : A_n \to A_m$ is injective for $m > n$. Let $C_{n.m}$ be the kernel of $j_{n.m}$.

In this paper, we will identify $C_{n.m}$ with a subgroup of B_n for $m \gg n >\gg 0$. This, in particular, will imply $A_n \simeq B_n$ for $n \gg 0$ as groups if Greenberg's conjecture holds. For this, we describe the structure of B_n in section 2. It is interesting that B_n behaves in an opposite way to A_n. Namely, for $m > n$, the norm map $N_{m,n} : A_m \to A_n$ is surjective, but the natural map $j_{n.m} : B_n \to B_m$ is injective. Moreover, the injectivity of the natural map $j_{n.m} : A_n \to A_m$ is equivalent to the surjectivity of the norm map $N_{m.n} : B_m \to B_n$. In section 3, we apply cohomological arguments to the structure of B_n to identify $C_{n.m}$ with a subgroup of B_n for $m \gg n \gg 0$.

2. Structure of A_n and B_n

The structure of A_n is well known [2]:

(1) $\varprojlim A_n \simeq \mathbf{Z}_p^\lambda \oplus C$,

(2) $A_n \simeq \oplus_{i=1}^\lambda \mathbf{Z}/p^{n+a_i}\mathbf{Z} \oplus C$ for $n \gg 0$,

(3) $\varinjlim A_n \simeq (\mathbf{Q}_p/\mathbf{Z}_p)^\lambda$,

for some finite group C and for some integers $a_1, a_2, \cdots, a_\lambda$, where λ is the Iwasawa λ-invariant for K_∞/K_0. Here the limits are taken under the norm maps (for (1)) and $j_{n.m}$ (for (3)). For $m \gg n \gg 0$, since $C \simeq C_{n.m}$ is the kernel of $j_{n.m}$, it disappears in the direct limit (3). On the other hand, since the norm map is surjective by class field theory, C remains in the inverse limit (1).

We analyse the structure of B_n similarly.

THEOREM 1:

(1)′ $\varprojlim B_n \simeq \mathbf{Z}_p^\lambda$,

(2)′ $B_n \simeq \oplus_{i=1}^\lambda \mathbf{Z}/p^{n+b_i}\mathbf{Z} \oplus M$,

(3)′ $\varinjlim B_n \simeq (\mathbf{Q}_p/\mathbf{Z}_p)^\lambda \oplus M$, *for some integers $b_1, b_2, \cdots, b_\lambda$ and for some finite group M, where λ is the Iwasawa λ-invariant.*

Proof: For $m > n \geq 0$, the natural map $j_{n.m} : B_n \to B_m$ is injective [1]. Let T be the torsion part of $\varprojlim B_n$. Then T is a finite group by the Iwasawa theory on the structure of Λ-modules. Hence if we take s large enough, Gal (K_∞/K_s) acts trivially on T. Suppose $X = (x_0, x_1, \dots)$ is an element of T, and so $p^N X = 0$ for some $N > 0$. For each $n > s$, take m such that $m - n > N$. Then

$$\begin{aligned} j_{n.m}(x_n) &= j_{n.m} \circ N_{m.n}(x_m) \\ &= x_m^{\sum \sigma}, \text{ where the summation is taken for all } \sigma \in \mathrm{Gal}(K_m/K_n) \\ &= x_m^{p^{m-n}} \text{ since each } \sigma \text{ acts trivially on } x \\ &= 0. \end{aligned}$$

Since $j_{n.m}$ is injective, we have $x_n = 0$. Since this is true for all $n > s$, X must be 0. Hence $T = 0$ and $\varprojlim B_n$ is torsion free. So $\varprojlim B_n \simeq \mathbf{Z}_p^r$ for some integer r. But, by the analytic class number formula or by the main conjecture which is now a theorem, $\varprojlim A_n$ and $\varprojlim B_n$ have the same $\mathbf{Z}_p$-rank. Therefore $r = \lambda =$ Iwasawa λ-invariant. This proves (1)$'$. (2)$'$ follows from the fact that $B_m^{G_{m.n}} = B_n$ which will be proved in section 3. Finally (3)$'$ follows from the injectivity of $j_{n.m}$ and (2)$'$.

REMARK: *From (1)$'$ and (2)$'$, we see that $N_{m.n} B_m \simeq \oplus_{i=1}^{\lambda} \mathbf{Z}/p^{n+b_i}\mathbf{Z} = B_n/M$ for $m \gg n \gg 0$.*

3. Main theorem

In section 2, we found two finite groups C and M: C from the structure of A_n and M from that of B_n. In this section we prove that they are isomorphic. First we list some known results about the (Tate) cohomology groups [1], [3].

(i) $C_m^{G_{m.n}} = C_n,\ E_m^{G_{m.n}} = E_n$

(ii) $H^0(G_{m.n}, C_m) = 0,\ H^1(G_{m.n}, C_m) \simeq \mathbf{Z}/p^{m-n}\mathbf{Z}$,

(iii) $H^0(G_{m.n}, E_m) = C',\ H^1(G_{m.n}, E_m) \simeq \mathbf{Z}/p^{m-n}\mathbf{Z} \oplus C$,

where C is the kernel of $j_{n.m}$ for $m >> n >> 0$ and C' is some finite group of the same order as C. Also, we have the following cohomology groups for B_m.

PROPOSITION: *For $m \gg n >\gg 0$, $H^1(G_{m.n}, B_m) \simeq M \simeq H^0(G_{m.n}, B_m)$.*

Proof: We claim that the map $H^1(G_{m.n}, C_m) \to H^1(G_{m.n}, E_m)$ induced by the inclusion $C_m \to E_m$ is injective. We have a generator $\pi_m^{\sigma-1}$ for $H^1(G_{m.n}, C_m)$, where $\pi_m = (1-\zeta_{p^{m+1}})(1-\zeta_{p^{m+1}}^{-1})$ and σ is a generator of $G_{m.n}$. Hence if $\pi_m^{\sigma-1}$ is mapped to the zero element of $H^1(G_{m.n}, E_m)$, then $\pi_m^{\sigma-1} = \eta^{\sigma-1}$ for some unit $\eta \in K_m$. Therefore we have $\pi_m = \eta\alpha_n$ for some $\alpha_n \in K_n$. As ideals, we have $(\pi_m) = (\alpha_n)$, which is impossible since (π_m) is the only totally ramified prime ideal in K_m above p. From the short exact sequence $0 \to C_m \to E_m \to B_m \to 0$, we get an exact sequence

$$\begin{array}{ccccccccc} 0 \to & C_m^{G_{m.n}} & \to & E_m^{G_{m.n}} & \to & B_m^{G_{m.n}} & \to & H^1(G_{m.n}, C_m) & \to & H^1(G_{m.n}, E_m) \\ & \downarrow \simeq & & \downarrow \simeq & & & & & & \\ & C_n & & E_n & & & & & & \end{array}$$

Since the last map is injective, we obtain an exact sequence

$$0 \to C_n \to E_n \to B_m^{G_{m.n}} \to 0.$$

Therefore $B_m^{G_{m.n}} \simeq E_n/C_n \simeq B_n$. On the other hand, by the remark at the end of section 2, $N_{m.n}B_m \simeq \oplus_{i=1}^{\lambda} \mathbf{Z}/p^{n+b_i}\mathbf{Z}$. Hence $H^0(G_{m.n}, B_m) \simeq M$. Since B_m is a finite group, the Herbrand quotient for B_m is 1, so $H^1(G_{m.n}, B_m)$ has the same order as M. But since $G_{m.n}$ acts trivially on M for $n >> 0$, M is a subgroup of $H^1(G_{m.n}, B_m)$. Therefore $H^1(G_{m.n}, B_m) \simeq M$.

Theorem 2: $C \simeq M$.

Proof: From the exact sequence $0 \to C_m \to E_m \to B_m \to 0$ and from the argument in the proof of the previous proposition, we have

$$0 \to H^1(G_{m.n}, C_m) \to H^1(G_{m.n}, E_m) \to H^1(G_{m.n}, B_m) \to 0.$$

By taking the direct limit under the inflation maps, we obtain

$$0 \to \mathbf{Q}_p/\mathbf{Z}_p \to \mathbf{Q}_p/\mathbf{Z}_p \oplus C \to M \to 0.$$

But a homomorphism from $\mathbf{Q}_p/\mathbf{Z}_p$ to a finite group must be a zero map. Hence we conclude $C \simeq M$.

Corollary: *$\sum_{i=1}^{\lambda} a_i = \sum_{i=1}^{\lambda} b_i$, where a_i's and b_i's are the integers appearing in section 2.*

Proof: It follows immediately from Theorem 2 and the fact that $^{\#}(A_n) = {}^{\#}(B_n)$ for all $n \geq 0$.

COROLLARY: *Suppose $\lambda \leq 1$, then $A_n \simeq B_n$ for $n >> 0$.*

Proof: Obvious.

REMARK: *It is unknown whether or not $A_n \simeq B_n$ in general. The last corollary answers this problem affirmatively in special cases. In particular, if Greenberg's conjecture is true, then $A_n \simeq B_n$ for sufficiently large n. The authors do not know whether $A_n \simeq B_n$ for all n even assuming the Greenberg's conjecture. One can also ask for generalizations of our results to the case when the base field K_0 is replaced by $\mathbf{Q}(\zeta_{pd})^+$ with $p \nmid d$. For this direction [4] might be helpful.*

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